

# MEASUREMENT-INDUCED NONLOCALITY OVER TWO-SIDED PROJECTIVE MEASUREMENTS

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**ABSTRACT.** Measurement-induced nonlocality (MiN), introduced by Luo and Fu [Phys. Rev. Lett. 106(2011)120401], is a kind of quantum correlation that beyond entanglement and even beyond quantum discord. Recently, we extended MiN to infinite-dimensional bipartite system [arXiv:1107.0355]. MiN is defined over one-sided projective measurements. In this letter we introduce a measurement-induced nonlocality over two-sided projective measurements. The nullity of this two-sided MiN is characterized, a formula for calculating two-sided MiN for pure states is proposed, and a lower bound of (two-sided) MiN for maximally entangled mixed states is given. In addition, we find that (two-sided) MiN is not continuous. The two-sided geometric measure of quantum discord (GMQD) is introduced in [Phys. Lett. A 376(2012)320–324]. We extend it to infinite-dimensional system and then compare it with the two-sided MiN. Both finite- and infinite-dimensional cases are considered.

## 1. INTRODUCTION

Quantum correlation in composite quantum states lies at the heart of quantum world [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Recently, much interest has been devoted to the study of quantum correlations that may arise without entanglement, such as quantum discord (QD) [16], measurement-induced nonlocality (MiN) [17] and quantum deficit [20], etc. These quantum correlations are argued to be responsible for certain quantum computation and quantum information tasks [4, 17, 21, 22, 23].

MiN is one of the ways to detect quantum correlation by locally invariant projective measurements [17]. Locally invariant measurements can not affect global states in classical theory but this is possible in quantum theory. So MiN is a type of quantum correlation which can only exist in quantum domain. It is indicated in [17] that MiN may be applied in quantum cryptography, general quantum dense coding [24, 25], remote state control [26, 27], etc.

Recently, the researching on MiN is arising: the monogamy of MiN is discussed [28]; in [29], a formula of MiN for arbitrary (finite) dimensional system is proposed, the dynamics of MiN is explored in [30]; MiN based on the relative entropy is discussed in [31]; in [32], we extended it to infinite-dimensional case and the nullity of MiN is characterized, etc. Xu in [33] introduced a geometric measure of quantum discord over two-sided projective measurements. In the same

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spirit, in this letter we introduce the two-sided MiN for both finite- and infinite-dimensional cases.

The paper is arranged as follows. In Sec. II, we first introduce the origin MiN, namely, the MiN over one-sided projective measurements. Then introduce the geometric measure of quantum discord (GMQD) proposed in [21, 32]. Sec. III is devoted to establishing the MiN over two-sided projective measurements for both finite- and infinite-dimensional cases. Then in Sec. IV we give a necessary and sufficient condition of a state has zero two-sided MiN. In Sec. V, we introduce the two-sided GMQD for infinite-dimensional case and compare the two-sided MiN with that of GMQD. A summarizing is given in the last section.

In this letter, we consider the two-mode system labeled by A+B which is described by a complex Hilbert space  $H = H_A \otimes H_B$  with  $\dim H_A \otimes H_B \leq +\infty$ . We denote by  $\mathcal{S}(H_A \otimes H_B)$  the set of all states acting on  $H_A \otimes H_B$ .

## 2. MEASUREMENT-INDUCED NONLOCALITY OVER ONE-SIDED PROJECTIVE MEASUREMENTS

Before discussing the two-sided MiN after this section, we firstly review the origin one-sided MiN in the present section.

MiN was firstly proposed by Luo and Fu [17], which can be viewed as a kind of quantum correlation from a geometric perspective based on the local projective measurements from which one of the reduced states is left invariant. The MiN of  $\rho$  (with respect to part A), denoted by  $N_A(\rho)$ , is defined by [17]

$$(1) \quad N_A(\rho) := \max_{\Pi^a} \|\rho - \Pi^a(\rho)\|_2^2,$$

where  $\|\cdot\|_2$  stands for the Hilbert-Schmidt norm (that is  $\|A\|_2 = [\text{Tr}(A^\dagger A)]^{\frac{1}{2}}$ ), and the maximum is taken over all local projective measurement  $\Pi^a = \{\Pi_k^a\}$  with  $\sum_k \Pi_k^a \rho_A \Pi_k^a = \rho_A$ , where  $\Pi^a(\rho) = \sum_k (\Pi_k^a \otimes I_B) \rho (\Pi_k^a \otimes I_B)$ .

Recently, we extended MiN to infinite-dimensional case [32] using the same scenario as that of finite-dimensional case. That is,

$$(2) \quad N_A(\rho) := \sup_{\Pi^a} \|\rho - \Pi^a(\rho)\|_2^2,$$

where the sup is taken over all local projective measurement  $\Pi^a = \{\Pi_k^a\}$  that satisfying  $\sum_k \Pi_k^a \rho_A \Pi_k^a = \rho_A$ . Here,  $\sum_k (\Pi_k^a \otimes I_B)^\dagger (\Pi_k^a \otimes I_B) = \sum_k \Pi_k^a \otimes I_B = I_{AB}$ , and the series converges under the strongly operator topology [19].

The MiN is in some sense dual to the geometric measure of quantum discord (GMQD) [17]. GMQD is originally introduced in [21] as

$$(3) \quad D_A^G(\rho) := \min_{\chi} \|\rho - \chi\|_2^2$$

with  $\chi$  runs over all zero QD (up to part A) states. Very recently, we extend GMQD to infinite-dimensional case in terms of the classical-quantum (CQ) states. [Let  $H_A$  and  $H_B$  are complex separable Hilbert spaces with  $\dim H_A \otimes H_B \leq +\infty$ . Recall that a state  $\rho$  is called a CQ state if it can be written as

$$(4) \quad \rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^B,$$

where  $\{|i\rangle\}$  is an orthonormal set of  $H_A$ ,  $\rho_i^B$ s are states of the subsystem B,  $p_i \geq 0$  and  $\sum_i p_i = 1$  (for the finite-dimensional case, see in [34]). Assume that  $\dim H_A \otimes H_B = +\infty$ , we define the geometric measure of quantum discord up to part A of a state by [32]

$$(5) \quad D_A^G(\rho) = \inf\{\|\rho - \pi\|_2^2 : \pi \in \mathcal{CQ}\},$$

where  $\mathcal{CQ}$  is the set of all CQ states on  $H_A \otimes H_B$ . That is, the geometric quantum discord of a state  $\rho$  is the square of the Hilbert-Schmidt distance of the state to the set of all CQ states.  $D_A^G(\rho)$  makes sense for any state  $\rho$  because states are Hilbert-Schmidt operators.

### 3. MEASUREMENT-INDUCED NONLOCALITY OVER TWO-SIDED PROJECTIVE MEASUREMENTS

We now define the MiN over two-sided projective measurements. We call  $\Pi^{ab} = \{\Pi^{\alpha\beta}\}$  a two-sided projective measurement if  $\Pi^{\alpha\beta} = |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$ , where  $\{|\alpha\rangle\}$  and  $\{|\beta\rangle\}$  are orthonormal bases of  $H_A$  and  $H_B$  respectively, and where  $\sum_{\alpha,\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| = I_A \otimes I_B$ , the series converges under the strong operator topology [19]. Under the operation of  $\Pi^{ab}$ ,  $\rho$  becomes

$$(6) \quad \Pi^{ab}(\rho) = \sum_{\alpha,\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \rho |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|.$$

A natural way to define two-sided MiN then is

$$(7) \quad N_{AB}(\rho) := \sup_{\Pi^{ab}} \|\rho - \Pi^{ab}(\rho)\|_2^2,$$

where the sup is taken over all two-sided projective measurements  $\{\Pi^{ab}\}$  that satisfying

$$\sum_{\alpha} |\alpha\rangle\langle\alpha| \rho_A |\alpha\rangle\langle\alpha| = \rho_A, \quad \sum_{\beta} |\beta\rangle\langle\beta| \rho_B |\beta\rangle\langle\beta| = \rho_B.$$

If  $\Pi^a = \{|\alpha\rangle\langle\alpha|\}$  is a local projective measurement on part A, and  $\Pi^b = \{|\beta\rangle\langle\beta|\}$  is a local projective measurement on part B, then  $\Pi^{ab} = \Pi^a \Pi^b = \Pi^b \Pi^a$  is a two-sided projective measurement and vice versa.

For the one-sided MiN, it is originally pointed out in [17] that, *it is particularly relevant to certain cryptographic communication*. The two-sided MiN may be also useful in some quantum information tasks. For example, consider the task that Alice and Bob are far away from each other and they want to send information to each other. Assume that they share a joint state  $\rho$ . Both Alice and Bob can encode their information by locally manipulating her/his part of the state, and then sends it to each other, they then decodes the message from the joint state. If Charlie and Dave are eavesdroppers that snoop around Alice and Bob respectively. In order to exclude eavesdropping in the communication processing, Alice and Bob choose measurements that will not disturb her/his local state  $\rho_{A/B}$ . Provided that  $N_{AB}(\rho) \neq 0$ , consequently, Alice and Bob can choose a measurement which maximizes the difference between the pre- and post-measurement states in order for both of them to detect the change of the joint state (thus the encoded messages) most reliably. Charlie and Dave can not get any information since Charlie is always facing the same state  $\rho_A$  and Dave is always facing the same state  $\rho_B$ , however  $\rho_A$  and  $\rho_B$  are invariant. In such a scenario, Alice and Bob can realize cryptographic communication successfully.

According to the definition in Eq. (7), the following properties are straightforward.

(i)  $N_{AB}(\rho) = 0$  if and only if  $N_A(\rho) = 0$  and  $N_B(\rho) = 0$ . Thus,  $N_{AB}(\rho) = 0$  for any product state  $\rho = \rho_A \otimes \rho_B$ , and a pure state has nonzero two-sided MiN if and only if it is entangled.

(ii)  $N_{AB}(\rho)$  is locally unitary invariant, namely,  $N_{AB}[(U \otimes V)\rho(U^\dagger \otimes V^\dagger)] = N_{AB}(\rho)$  for any unitary operators  $U$  and  $V$  acting on  $H_A$  and  $H_B$ , respectively.

(iii)  $N_{AB}(\rho) > 0$  whenever  $\rho$  is entangled since  $\Pi^{ab}(\rho)$  is always a classical state (the definition of classical state see in Sec. V) and thus is separable.

(iv)  $0 \leq N_{AB}(\rho) < 4$ .

(v) If both  $\rho_A$  and  $\rho_B$  are nondegenerate, then the two-sided projective measurement that make  $\rho_{A/B}$  invariant is unique, and this unique one is just the one induced by the eigenvectors of  $\rho_A$  and  $\rho_B$ .

To evaluate the two-sided MiN is a hard work in general, but it is easy for pure states.

**Theorem 1.** Let  $H_A$  and  $H_B$  be complex separable Hilbert spaces with  $\dim H_A \otimes H_B \leq +\infty$ , and let  $|\psi\rangle \in H_A \otimes H_B$  be a pure state with the Schmidt decomposition  $|\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle$ . Then

$$(8) \quad N_{AB}(|\psi\rangle) = N_A(|\psi\rangle) = N_B(|\psi\rangle) = 1 - \sum_k \lambda_k^4 \leq 1.$$

**Proof.** If  $|\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle$  is the Schmidt decomposition of  $|\psi\rangle$ , then, for any local projective measurement  $\Pi^{ab}$  that leaves  $\rho_A$  and  $\rho_B$  invariant, one has

$$\begin{aligned} & \Pi^{ab}(|\psi\rangle\langle\psi|) \\ &= \sum_{\lambda_k \neq \lambda_l} \lambda_k^2 |k\rangle\langle k| \otimes |k'\rangle\langle k'| \\ &+ \sum_{\lambda_k = \lambda_l} \lambda_k \lambda_l \left( \sum_i \alpha_{ki} \bar{\alpha}_{li} |e_i\rangle\langle e_i| \right) \otimes \left( \sum_j \beta_{kj} \bar{\beta}_{lj} |e'_j\rangle\langle e'_j| \right), \end{aligned}$$

where  $\{|e_i\rangle\}$  (resp.  $\{|e'_j\rangle\}$ ) satisfy  $\sum_i |e_i\rangle\langle e_i| = I_{E(\lambda_k)}$  (resp.  $\sum_j |e'_j\rangle\langle e'_j| = I_{E'(\lambda_k)}$ ) whenever  $\lambda_k = \lambda_l$ , here  $E(\lambda_k)$  (resp.  $E'(\lambda_k)$ ) denotes the eigenspace of eigenvalue  $\lambda_k^2$  of  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$  (resp.  $\rho_B = \text{Tr}_A(|\psi\rangle\langle\psi|)$ ), and where  $\alpha_{ki} = \langle e_i | k \rangle$ ,  $\beta_{lj} = \langle e'_j | l' \rangle$ . We compute

$$\begin{aligned} & \| |\psi\rangle\langle\psi| - \Pi^{ab}(|\psi\rangle\langle\psi|) \|_2^2 \\ &= \left\| \sum_{k,l} \lambda_k \lambda_l |k\rangle\langle l| \otimes |k'\rangle\langle l'| - \sum_{\lambda_k \neq \lambda_l} \lambda_k^2 |k\rangle\langle k| \otimes |k'\rangle\langle k'| \right. \\ &\quad \left. - \sum_{\lambda_k = \lambda_l} \lambda_k \lambda_l \left( \sum_i \alpha_{ki} \bar{\alpha}_{li} |e_i\rangle\langle e_i| \right) \otimes \left( \sum_j \beta_{kj} \bar{\beta}_{lj} |e'_j\rangle\langle e'_j| \right) \right\|_2^2 \\ &= \text{Tr} \left[ \left( \sum_{k,l} \lambda_k \lambda_l |k\rangle\langle l| \otimes |k'\rangle\langle l'| - \sum_{\lambda_k \neq \lambda_l} \lambda_k^2 |k\rangle\langle k| \otimes |k'\rangle\langle k'| \right. \right. \\ &\quad \left. \left. - \sum_{\lambda_k = \lambda_l} \lambda_k \lambda_l \left( \sum_i \alpha_{ki} \bar{\alpha}_{li} |e_i\rangle\langle e_i| \right) \otimes \left( \sum_j \beta_{kj} \bar{\beta}_{lj} |e'_j\rangle\langle e'_j| \right) \right)^2 \right] \\ &= 1 + \sum_k \lambda_k^4 - 2 \sum_{\lambda_k \neq \lambda_l} \lambda_k^4 - 2 \sum_{\lambda_k = \lambda_l} \lambda_k^2 \lambda_l^2 \\ &= 1 - \sum_k \lambda_k^4. \end{aligned}$$

Together with the fact [17, 32]  $N_A(|\psi\rangle) = N_B(|\psi\rangle) = 1 - \sum_k \lambda_k^4$ , we complete the proof.  $\square$

Theorem 1 implies that  $N_{AB}$  achieves the maximal value  $\frac{m-1}{m}$  for maximally entangled pure state in  $m \otimes n$  system with  $m \leq n$ . Next, we consider the (two-sided) MiN for maximally

entangled mixed states. It is known that, for  $m \otimes n$  ( $m \leq n$ ) system, a pure state  $|\psi\rangle$  is called maximally entangled if  $\rho_A = \frac{1}{m}I_A$ . If a mixed state has the same entanglement quantified by a certain entanglement measure as  $|\psi\rangle$ , we call it a maximally entangled mixed state [35].

**Theorem 2.** Assume that  $\dim H_A = m$ ,  $\dim H_B = n$  and  $2m \leq n$ . If  $\rho$  is a maximally entangled mixed state on  $H_A \otimes H_B$ , then

$$(9) \quad N_A(\rho) = \frac{m-1}{m}, \quad N_B(\rho) \geq \frac{m-1}{m}, \quad N_{AB}(\rho) \geq \frac{m-1}{m}.$$

**Proof.** Let  $\rho$  be a maximally entangled mixed state. By Theorem 1 in [35],  $\rho$  can be written as

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad \sum_k p_k = 1,$$

where  $|\psi_k\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |i\rangle|i'_k\rangle$ ,  $\{|i\rangle\}$  and  $\{|i'_k\rangle\}$  are orthonormal bases of the subsystems A and B respectively, satisfying  $\langle j_s|i_t\rangle = \delta_{ij}\delta_{st}$ . For simplicity, we write  $\rho_A^{(i)} = \text{Tr}_B(|\psi_i\rangle\langle\psi_i|)$ . Obviously,  $\rho_A = \rho_A^{(i)} = \frac{1}{m}I_A$  for any  $i$ , which reveals that any local projective  $\Pi^a$  make  $\rho_A$  invariant, and thus make  $\rho_A^{(i)}$  invariant. Notice that for any local projective measurement  $\Pi^a$ , we have

$$\text{Tr}(\Pi^a(|\psi_i\rangle\langle\psi_i|)\Pi^a(|\psi_j\rangle\langle\psi_j|)) = 0$$

whenever  $i \neq j$ . It follows that

$$\begin{aligned} & \|\rho - \Pi^a(\rho)\|_2 \\ &= \left\| \sum_k p_k (|\psi_k\rangle\langle\psi_k| - \Pi^a(|\psi_k\rangle\langle\psi_k|)) \right\|_2 \\ &= \sum_k p_k \| |\psi_k\rangle\langle\psi_k| - \Pi^a(|\psi_k\rangle\langle\psi_k|) \|_2 \\ &= \sum_k p_k \sqrt{\frac{m-1}{m}} \\ &= \sqrt{\frac{m-1}{m}} \end{aligned}$$

holds for any local projective measurement  $\Pi^a$ , that is  $N_A(\rho) = \frac{m-1}{m}$  as desired.

Let  $\rho_B^{(i)} = \text{Tr}_A(|\psi_i\rangle\langle\psi_i|)$ . For any local projective measurement  $\Pi^b$  with  $\Pi^b(\rho_B^{(i)}) = \rho_B^{(i)}$ , we have

$$\text{Tr}(\Pi^b(|\psi_i\rangle\langle\psi_i|)\Pi^b(|\psi_j\rangle\langle\psi_j|)) = 0$$

whenever  $i \neq j$  and  $\Pi^b(\rho_B) = \rho_B$ . We assume further that  $N_B(|\psi_k\rangle) = \||\psi_k\rangle\langle\psi_k| - \Pi^b(|\psi_k\rangle\langle\psi_k|)\|_2^2$ . Then

$$\begin{aligned}
& \|\rho - \Pi^b(\rho)\|_2 \\
&= \left\| \sum_k p_k (|\psi_k\rangle\langle\psi_k| - \Pi^b(|\psi_k\rangle\langle\psi_k|)) \right\|_2 \\
&= \sum_k p_k \||\psi_k\rangle\langle\psi_k| - \Pi^b(|\psi_k\rangle\langle\psi_k|)\|_2 \\
&= \sum_k p_k \sqrt{\frac{m-1}{m}} \\
&= \sqrt{\frac{m-1}{m}},
\end{aligned}$$

which leads to  $N_B(\rho) \geq \frac{m-1}{m}$ .

Replace  $\Pi^b$  by  $\Pi^{ab}$  and repeat the argument above, we can conclude that  $N_{AB}(\rho) \geq \frac{m-1}{m}$ .  $\square$

**Questions.** Is it true if we replace ' $\geq$ ' by ' $=$ ' in Eq. (9)? Is  $N_{A/B}$  (or  $N_{AB}$ ) always smaller than  $\frac{m-1}{m}$  for any  $m \otimes n$  system with  $m \leq n$ ?

By definition, it is obvious that  $D_{A/B}^G$  and  $D_{AB}^G$  are continuous functions on  $\mathcal{S}(H_A \otimes H_B)$  with respect to trace norm (note that the trace norm topology coincides with the Hilbert-Schmidt norm topology on  $\mathcal{S}(H_A \otimes H_B)$  [36]). In the end of this section, we show that

**Proposition 3.**  $N_{A/B}$  and  $N_{AB}$  are not continuous functions on  $\mathcal{S}(H_A \otimes H_B)$  with respect to the trace norm ( $\dim H_A \otimes H_B \leq +\infty$ ).

We illustrate this fact with the example below. We consider the  $m \otimes m$  Werner state

$$(10) \quad \rho_x = \frac{m-x}{m^3-m} I_A \otimes I_B + \frac{mx-1}{m^3-m} F, \quad x \in [0, 1],$$

with  $F = \sum_{i,j} |i\rangle\langle j| \otimes |j'\rangle\langle i'|$  is the flip operator. Let  $\sigma = (\sum_{i=1}^m \lambda_i |i\rangle\langle i|) \otimes (\sum_{j=1}^m \delta_j |j'\rangle\langle j'|)$  be a  $m \otimes m$  state with different  $\lambda_i$ s and  $\delta_j$ s such that the reduced states  $\sigma_A$  and  $\sigma_B$  are nondegenerate. In contrast, let  $\varrho = (\sum_{i=1}^m \lambda_i |i\rangle\langle i|) \otimes (\sum_{j=1}^m \delta_j |\mu'_j\rangle\langle \mu'_j|)$  with  $\{|\mu'_j\rangle\}$  is another orthonormal basis of  $H_B$  such that  $\{|j'\rangle\}$  and  $\{|\mu'_j\rangle\}$  are unbiased. Let  $\epsilon_n$  and  $\varepsilon_n$  be infinitesimal numbers. Write  $\rho_n = \frac{1}{1+\epsilon}(\rho_x + \epsilon_n \sigma)$  and  $\varrho_n = \frac{1}{1+\epsilon_n}(\rho_x + \varepsilon_n \varrho)$ . Then the unique two-sided projective measurement  $\Pi^{ab}$  on  $\rho_n$  that leaves both  $\rho_A^{(n)}$  and  $\rho_B^{(n)}$  invariant is the one that induced from the eigenvector of  $\rho_A^{(n)}$  and  $\rho_B^{(n)}$ , and the unique two-sided projective measurement  $\Pi'^{ab}$  on  $\varrho_n$  that leaves both  $\varrho_A^{(n)}$  and  $\varrho_B^{(n)}$  invariant is the one that induced from the eigenvector of  $\varrho_A^{(n)}$  and  $\varrho_B^{(n)}$  (here  $\rho_{A/B}^{(n)} = \text{Tr}_{B/A}(\rho_n)$ ,  $\varrho_{A/B}^{(n)} = \text{Tr}_{B/A}(\varrho_n)$ ). Note that  $\rho_A^{(n)}$ s (resp.  $\varrho_A^{(n)}$ s) have the same eigenvectors, and  $\rho_B^{(n)}$ s (resp.  $\varrho_B^{(n)}$ s) have the same eigenvectors as well). It is now clear that, although  $\rho_n \rightarrow \rho_x$  and  $\varrho_n \rightarrow \rho_x$  in the trace norm for some sequences  $\{\epsilon_n\}$  and  $\{\varepsilon_n\}$  (and thus  $\varrho_n - \rho_n \rightarrow 0$  in the trace norm),  $\|\varrho_n - \Pi'^{ab}(\varrho_n)\|_2 - \|\rho_n - \Pi^{ab}(\rho_n)\|_2 \not\rightarrow 0$ , which implies that  $N_{AB}(\varrho_n) - N_{AB}(\rho_n) \not\rightarrow 0$ . Therefore,  $N_{AB}$  is not continuous. Similarly, one can check that  $N_{A/B}$  is not continuous.

## 4. NULLITY OF TWO-SIDED MEASUREMENT-INDUCED NONLOCALITY

Let  $H_A$  and  $H_B$  be complex separable Hilbert spaces with  $\dim H_A \otimes H_B \leq +\infty$ , and let  $\{|i\rangle\}$  and  $\{|i'\rangle\}$  be the orthonormal bases of  $H_A$  and  $H_B$  respectively. Then any state  $\rho$  acting on  $H_A \otimes H_B$  can be represented by

$$(11) \quad \rho = \sum_{i,j} E_{ij} \otimes B_{ij} = \sum_{k,l} A_{kl} \otimes F_{kl},$$

where  $E_{ij} = |i\rangle\langle j|$ ,  $F_{kl} = |k'\rangle\langle l'|$ ,  $B_{ij}$ s and  $A_{kl}$ s are trace class operators on  $H_B$  and  $H_A$  respectively, and where the series converges in trace norm [37].

In [32], we proved that  $N_A(\rho) = 0$  if and only if  $A_{kl}$ s are mutually commuting normal operators and each eigenspace of  $\rho_A$  contained in some eigenspace of  $B_{ij}$  for all  $k$  and  $l$ . Equivalently, we also obtain in [32] that  $N_A(\rho) = 0$  if and only if

$$(12) \quad \rho = \sum_j p_j |j\rangle\langle j| \otimes \rho_j^B$$

with  $\rho_j^B = \rho_i^B$  whenever  $p_j = p_i$ , where  $\{|j\rangle\}$  is an orthonormal basis of  $H_A$ . Together with the fact  $N_{AB}(\rho) = 0$  if and only if  $N_A(\rho) = N_B(\rho) = 0$ , the followings are obvious.

**Theorem 4.** Let  $H_A$  and  $H_B$  be complex separable Hilbert spaces with  $\dim H_A \otimes H_B \leq +\infty$ ,  $\{|k\rangle\}$  and  $\{|i'\rangle\}$  be orthonormal bases of  $H_A$  and  $H_B$ , respectively, and  $\rho \in \mathcal{S}(H_A \otimes H_B)$ . Write  $\rho = \sum_{i,j} E_{ij} \otimes B_{ij} = \sum_{k,l} A_{kl} \otimes F_{kl}$  as in Eq. (11) with respect to the given bases. Then  $N_{AB}(\rho) = 0$  if and only if  $B_{ij}$ s and  $A_{kl}$ s are mutually commuting normal operators with the properties that each eigenspace of  $\rho_B$  (resp.  $A_{kl}$ ) contained in some eigenspace of  $B_{ij}$  ( $A_{kl}$ ) for all  $i$  and  $j$  (resp.  $k$  and  $l$ ).

Equivalently, we have

**Theorem 5.** Let  $H_A$  and  $H_B$  be complex separable Hilbert spaces with  $\dim H_A \otimes H_B \leq +\infty$ ,  $\rho \in \mathcal{S}(H_A \otimes H_B)$ . Then  $N_{AB}(\rho) = 0$  if and only if

$$(13) \quad \rho = \sum_{i,j} p_{i,j} |i\rangle\langle i| \otimes |j'\rangle\langle j'|$$

with  $p_{ij} = p_{ik}$  whenever  $\sum_i p_{ij} = \sum_i p_{ik}$  and  $p_{ij} = p_{lj}$  whenever  $\sum_j p_{ij} = \sum_j p_{lj}$ .

Assume that  $\dim H_A \otimes H_B < +\infty$ ,  $\dim H_A = m$  and  $\dim H_B = n$ . Let  $\rho \in \mathcal{S}(H_A \otimes H_B)$  be a state with  $N_{AB}(\rho) = 0$ . Theorem 4 and 5 reveals that

$$(14) \quad N_{AB}(t\rho + \frac{1-t}{mn} I_{AB}) = 0, \quad t \in [0, 1]$$

and

$$(15) \quad N_{AB}(\rho^{T_{A/B}}) = 0,$$

where  $\rho^{T_{A/B}}$  denotes the partial transpose of  $\rho$ . One can also check that if  $D_{AB}(\rho) = 0$ , then

$$D_{AB}(t\rho + \frac{1-t}{mn} I_{AB}) = 0, \quad t \in [0, 1]$$

and

$$D_{AB}(\rho^{T_{A/B}}) = 0.$$

Going further, (i) if  $N_{A/B}(\rho) = 0$ , then  $N_{A/B}(t\rho + \frac{1-t}{mn}I_{AB}) = 0$  for all  $t \in [0, 1]$  and  $N_{A/B}(\rho^{T_{A/B}}) = 0$ ; (ii) if  $D_{A/B}(\rho) = 0$ , then  $D_{A/B}(t\rho + \frac{1-t}{mn}I_{AB}) = 0$  for all  $t \in [0, 1]$  and  $D_{A/B}(\rho^{T_{A/B}}) = 0$ . That is, both the nullity of (two-sided) MiN and the nullity of (two-sided) QD are connected sets since all zero (two-sided) MiN states and all zero (two-sided) QD states are connected with the maximal mixed state of the system. And the partial transpose of zero (two-sided) MiN (QD) state is still a zero (two-sided) MiN (QD) state.

## 5. COMPARING WITH (GEOMETRIC MEASURE OF) QUANTUM DISCORD

We now begin to discuss the relation between the two-sided MiN and the two-sided QD (GMQD). For the finite-dimensional case, the QD over two-sided projective measurements is defined by [16, 33]

$$(16) \quad D_{AB}(\rho) = I(\rho) - \sup_{\Pi^{ab}} I(\Pi^{ab}(\rho)),$$

where sup is taken over all two-sided projective measurements. It is known that  $D_{AB}(\rho) = 0$  if and only if  $\rho$  is a *classical* state (see in [33] and the references therein). Recall that, a bipartite state  $\rho$  acting on  $H_A \otimes H_B$  with  $\dim H_A \otimes H_B \leq +\infty$  is called a classical state, if it admits the form of

$$(17) \quad \rho = \sum_{i,j} p_{ij} |i\rangle\langle i| \otimes |j'\rangle\langle j'|,$$

where  $\sum_{i,j} p_{ij} = 1$ ,  $p_{ij} \geq 0$ . For finite-dimensional system, the GMQD over two-sided projective measurements is defined by [33]

$$(18) \quad D_{AB}^G(\rho) = \inf_{\pi} \|\rho - \pi\|_2^2,$$

where inf is taken over all  $\pi$  that  $D_{AB}(\pi) = 0$ . Let  $\mathcal{C}$  be the set of all classical states in  $\mathcal{S}(H_A \otimes H_B)$ . We now extend it to infinite-dimensional case by

$$(19) \quad D_{AB}^G(\rho) = \inf_{\varrho} \|\rho - \varrho\|_2^2,$$

where inf is taken over all  $\varrho \in \mathcal{C}$ .

Denote by  $\mathcal{N}_{AB}^0$ ,  $\mathcal{D}_{AB}^0$  and  $\mathcal{D}_{AB}^{G-0}$  the nullity of two-sided MiN, the nullity of two-sided QD and the nullity of two-sided GMQD, respectively. We showed in [32] that  $D_A^G(\rho) = 0$  if and only if  $\rho$  is a CQ state for both finite- and infinite-dimensional cases. Hence  $\mathcal{D}_{AB}^0 = \mathcal{D}_{AB}^{G-0} = \mathcal{C}$  holds for finite-dimensional case and  $\mathcal{D}_{AB}^{G-0} = \mathcal{C}$  holds for infinite-dimensional case. Let  $\mathcal{N}_A^0$  (resp.  $\mathcal{N}_B^0$ ) be the set of all states with zero MiN with respect to part A (resp. B). Recall that, a state  $\rho \in \mathcal{S}(H_A \otimes H_B)$  is called a quantum-classical (QC) state, if it can be expressed by  $\rho = \sum_j q_j \rho_j^A \otimes |j'\rangle\langle j'|$  for some orthonormal basis  $\{|j'\rangle\}$  of  $H_B$ , where  $\{q_j\}$  is a probability distribution,  $\rho_j^A$ 's are states on  $H_A$ . Let  $\mathcal{QC}$  be the set of all QC states, then  $\mathcal{C} = \mathcal{CQ} \cap \mathcal{QC}$ . In [32], we proved that  $\mathcal{N}_A^0$  (resp.  $\mathcal{N}_B^0$ ) is a proper subset of  $\mathcal{CQ}$  (resp.  $\mathcal{QC}$ ). Together with Theorem 5, we can thus get that



**Theorem 6.**  $\mathcal{N}_{AB}^0$  is a proper subset of  $\mathcal{D}_{AB}^{G-0}$  for both finite- and infinite-dimensional cases.

For finite-dimensional case, analytic formulas for calculating the GMQD and the two-side GMQD are proposed [18, 33]. Let  $\rho \in \mathcal{S}(H_A \otimes H_B)$  with  $\dim H_A \otimes H_B < +\infty$ . Assume that  $\dim H_A = m$  and  $\dim H_B = n$ . We denote by  $\mathcal{B}(H_{A/B})$  the space of all linear operators on  $H_{A/B}$ . For given sets of Hermitian operators  $\{X_i : i = 1, 2, \dots, m^2\}$  and  $\{Y_j : j = 1, 2, \dots, n^2\}$  that constitute orthonormal Hilbert-Schmidt bases of  $\mathcal{B}(H_A)$  and  $\mathcal{B}(H_B)$  respectively,  $\rho$  can be decomposed as  $\rho = c_{ij}X_i \otimes Y_j$ . For any orthonormal bases  $\{|k\rangle\}$  of  $H_A$  and  $\{|l'\rangle\}$  of  $H_B$ , let  $|k\rangle\langle k| = \sum_{i=1}^{m^2} a_{ki}X_i$ ,  $|l'\rangle\langle l'| = \sum_{j=1}^{n^2} b_{lj}Y_j$ ,  $A = [a_{ki}]$ ,  $B = [b_{lj}]$  and  $C = [c_{ij}]$ . It is showed in [18] that

$$D_A^G(\rho) = \text{Tr}(CC^t) - \sup_{A,B} \text{Tr}(ACC^tA^t),$$

$$D_A^G(\rho) \geq \text{Tr}(CC^t) - \sum_{k=1}^m \lambda_k.$$

In [33], the  $D_{AB}^G(\rho)$  is evaluated by

$$D_{AB}^G(\rho) = \text{Tr}(CC^t) - \sup_{A,B} \text{Tr}(ACB^tBC^tA^t),$$

$$D_{AB}^G(\rho) \geq \text{Tr}(CC^t) - \sum_{k=1}^{\min\{m,n\}} \lambda_k,$$

where  $\lambda_k$  are eigenvalues of  $CC^t$  listed in decreasing order (counting multiplicity),  $t$  denotes transpose.

If  $\rho_A = \frac{1}{m}I_A$ , then all local projective measurements  $\Pi^a$  leave  $\rho_A$  invariant. In such a case, it is immediate from Eqs. (1) and (3) that  $N_A(\rho) \geq D_A^G(\rho)$ . Symmetrically, if  $\rho_B = \frac{1}{n}I_B$ , one has  $N_B(\rho) \geq D_B^G(\rho)$ . And of course that,  $N_{AB}(\rho) \geq D_{AB}^G(\rho)$  provided that  $\rho_A = \frac{1}{m}I_A$  and  $\rho_B = \frac{1}{n}I_B$ . The following result is easy to check.

**Theorem 7.** Let  $\rho \in \mathcal{S}(H_A \otimes H_B)$  be a state with  $\dim H_A = m < +\infty$  and  $\dim H_B = n < +\infty$ . Then the following statements are true:

(i) If  $\rho_A = \frac{1}{m}I_A$ , then

$$N_A(\rho) = \text{Tr}(CC^t) - \inf_{A,B} \text{Tr}(ACC^tA^t) \geq D_A(\rho),$$

$$N_A(\rho) \geq D_A(\rho) \geq \text{Tr}(CC^t) - \sum_{k=1}^m \lambda_k;$$

(ii) If  $\rho_B = \frac{1}{n}I_B$ , then

$$N_B(\rho) = \text{Tr}(CC^t) - \inf_{A,B} \text{Tr}(CB^tBC^t) \geq D_B(\rho),$$

$$N_B(\rho) \geq D_B(\rho) \geq \text{Tr}(CC^t) - \sum_{k=1}^n \lambda_k;$$

(iii) If  $\rho_A = \frac{1}{m}I_A$  and  $\rho_B = \frac{1}{n}I_B$ , then

$$N_{AB}(\rho) = \text{Tr}(CC^t) - \inf_{A,B} \text{Tr}(ACB^tBC^tA^t) \geq D_{AB}(\rho),$$

$$N_{AB}(\rho) \geq D_{AB}(\rho) \geq \text{Tr}(CC^t) - \sum_{k=1}^{\min\{m,n\}} \lambda_k.$$

## 6. EXAMPLES

At last, we present some examples to illustrate the nullity of two-sided MiN and compare it with the two-sided QD(GMQD).

**Example 1.** We consider the two-qubit Bell-diagonal states

$$\rho = \frac{1}{4}(I_A \otimes I_B + \sum_{i=1}^3 \sigma_i \otimes \sigma_i),$$

where  $c_i$ s are real numbers,  $\sigma_i$ s are Pauli matrices, i.e.,  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$  and  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . It is showed in [33] that  $D_{AB}^G(\rho) = 0$  if and only if  $c_1 = c_2 = 0$ . However, one can check that  $N_{AB}(\rho) = 0$  if and only if  $c_1 = c_2 = c_3 = 0$ . Also note that  $\rho_A = \rho_B = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . That is  $D_{AB}^G(\rho) \geq N_{AB}(\rho)$  for Bell-diagonal state  $\rho$  and there exist Bell-diagonal states such that they have quantum correlation tested by two-sided QD while they don't contain quantum correlation quantified by two-sided MiN.

However, as what we will show, for the so-called Werner states and the isotropic states, the zero two-sided MiN coincides with the zero two-sided QD(GMQD).

**Example 2.** For the  $m \otimes m$  Werner state as in Eq. (10), it is showed that  $D_A^G(\rho_x) = 0$  if and only if  $x = \frac{1}{m}$  [18],  $D_{AB}^G(\rho_x) = 0$  if and only if  $x = \frac{1}{m}$  [33]. Clearly,  $N_{AB}(\rho_x) = 0$  if and only if  $x = \frac{1}{m}$ . Namely, the nullity of (two-sided) MiN coincides with that of (two-sided) QD(GMQD), which is the trivial one, i.e.,  $\rho$  is the maximally mixed state.

**Example 3.** For the  $m \otimes m$  isotropic state

$$\rho_x = \frac{1-x}{m^2-1}I_A \otimes I_B + \frac{m^2x-1}{m^2-1}|\psi^+\rangle\langle\psi^+|, \quad x \in [0, 1],$$

with  $|\psi^+\rangle$  is the maximally entangled state. It is known that  $D_A^G(\rho_x) = 0$  if and only if  $x = \frac{1}{m^2}$  [18] and  $D_{AB}^G(\rho_x) = 0$  if and only if  $x = \frac{1}{m^2}$  [33]. Apparently,  $N_{AB}(\rho_x) = 0$  if and only if  $x = \frac{1}{m^2}$ . So the nullity of (two-sided) MiN coincides with that of (two-sided) QD(GMQD).

## 7. SUMMARY

With the same spirit as the two-sided geometric measure of quantum discord introduced in [33], and in some sense dual to it, the two-sided measurement-induced nonlocality is established for both finite- and infinite-dimensional cases. Consequently, an analytic formula of  $N_{AB}$  for pure states is given, the nullity of the two-sided MiN is explored, and show that there exist nonzero two-sided MiN states that don't have two-sided QD. Furthermore, the two-sided GMQD for infinite-dimensional system is proposed, and then we compare it with (two-sided) MiN. We thus obtain a clear structure of these two different quantum correlations.

Generally, entanglement measure is continuous, for example, concurrence is such a situation [38], and the Negativity [39] is continuous as well. However, in contrast to it, we illustrate with example that both the origin one-sided MiN and the two-sided MiN are not continuous functions of the states.

Except for the questions proposed in Sec. III, there are some other issues that are worth further discussion. The first one is to comparing the dynamics of (two-sided) MiN with that of (two-sided) QD(GMQD). The second concerns the comparing of MiN with the well-accepted entanglement measure, entanglement of formation [40] or concurrence [38, 41]. These works will provide us a more explicit picture of these different quantum correlations.

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